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Formation of MACHO-Primordial Black Holes in Inflationary Cosmology

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Abstract

As a nonbaryonic explanation of massive compact halo objects, a phenomenological model is presented which predicts formation of primordial black holes at a desired mass scale. The required feature of initial density fluctuation is realized making use of the primordially isocurvature fluctuation generated in an inflationary universe model with multiple scalar fields.

1 Introduction

If overdensity of order of unity exists in the hot early universe, a black hole can be formed when the perturbed region enters the cosmological horizon. The primordial black holes (hereafter PBHs) thus produced was a subject of active research decades ago (Zel'dovich & Novikov 1967; Hawking 1971) and various observational constraints have been obtained against their mass spectrum —with no observational evidence of their existence at that time (Novikov et al. 1979).

Recently, however, several independent projects reported observation of massive compact halo objects (MACHOs) through gravitational microlensing (Alcock et al. 1993; Aubourg et al. 1993). It is estimated that their mass is around $0.01 - 0.1M_\odot$ and that they occupy $\sim 20\%$ of the galactic halo mass which makes up about $\mathcal{O}(10^{-3})$ of the critical density (Griest et al. 1995). While the primary candidate of MACHOs is substellar baryonic objects such as brown dwarfs, it is difficult to reconcile such a large amount of these objects with the observed mass function of low mass stars (Richer & Fahlman 1992) and with the infrared observation of dwarf component (Boughn & Uson 1995), unless the mass function is extrapolated to the lower masses in an extremely peculiar manner. Therefore it is also an interesting and potentially important theoretical issue to consider nonbaryonic explanation of the origin of MACHOs.

In the present paper we consider the possibility that MACHOs consist of PBHs produced in the early universe and present a simple model which generates a desired spectrum of primordial density perturbations in the context of inflationary cosmology (Guth 1981; Sato 1981; for a review see, *e.g.* Olive 1990). In the simplest models of inflation with one *inflaton* scalar field, the predicted adiabatic density fluctuation has an almost scale-invariant spectrum (Hawking 1982; Starobinsky 1982; Guth & Pi 1982), unless the inflaton has a peculiar potential. Hence they do not predict PBH formation in general. In models with multiple scalar fields, on the other hand, not only adiabatic but also isocurvature fluctuations are generated during inflation. The latter can be cosmologically important if energy density of its carrier becomes significant in a later epoch (Linde 1985; Kofman & Linde 1987). Furthermore it is relatively easier to imprint a nontrivial feature on the spectral shape of the isocurvature fluctuations. Making use of this property here we construct a model which possesses a peak in the spectrum of total density fluctuation at the horizon crossing. Then a significant amount of PBHs can be produced around the horizon mass scale when the fluctuation at the horizon crossing becomes maximal. While our goal is to produce PBHs of mass $\sim 0.1M_\odot$ with the

abundance of $\Omega_{\rm BH} \sim 10^{-3}$, one can easily see that our model can also be applied to PBH formation of different masses and abundances as well by choosing different values of model parameters.

The rest of the paper is organized as follows. In §2 we review basics of PBH formation and discuss necessary initial condition of density fluctuations to obtain adequate PBHs. Then in §3 possibility of generating the necessary fluctuations is considered in the context of inflationary cosmology and a model Lagrangian is proposed. §§4-6 are devoted to detailed description of the evolution of the universe in this model and in §7 constraints on the model parameters are obtained. §8 is the conclusion.

2 Formation processes of the PBHs.

PBHs are formed if initial density fluctuations grow sufficiently and a high density region collapses within its gravitational radius. First let us review its formation process. The background spacetime of the early universe dominated by radiation is satisfactorily described by the spatially-flat Friedmann universe,

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right], \tag{1}$$

whose expansion rate is given by

$$H^2(t) \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t), \tag{2}$$

with $\rho(t)$ being the background energy density and a dot denotes time derivation. Following Carr (1975), let us consider a spherically symmetric high density region with its initial radius, $R(t_0)$, larger than the horizon scale $\sim t_0$. The assumption of spherical symmetry will be justified below. Then the perturbed region locally constitutes a spatially closed Friedmann universe with a metric,

$$ds^2 = -dt'^2 + R^2(t')\left[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right], \qquad \kappa > 0, \tag{3}$$

Then the Einstein equation reads

$$H'^2(t') \equiv \left(\frac{1}{R}\frac{dR}{dt'}\right)^2 = \frac{8\pi G}{3}\rho_+(t') - \frac{\kappa}{R^2(t')}, \tag{4}$$

there, where ρ_+ is the local energy density. One can choose the coordinate so that both the background and the perturbed region have the same expansion rate

initially at $t = t' = t_0$. Then the initial density contrast, δ_0, satisfies

$$\delta_0 \equiv \frac{\rho_{+0} - \rho_0}{\rho_0} = \frac{\kappa}{H_0^2 R_0^2}, \tag{5}$$

where a subscript 0 implies values at t_0. It can be shown that the two time variables are related by (Harrison 1970)

$$(1+\delta_0)^{\frac{3}{4}} \frac{dt'}{R(t')} = \frac{dt}{a(t)}. \tag{6}$$

The perturbed region will eventually stop expanding at $R \equiv R_c$, which is obtained from

$$0 = \frac{8\pi G}{3}\rho_0(1+\delta_0)\left(\frac{R_0}{R_c}\right)^4 - \frac{\kappa}{R_c^2} = H_0^2(1+\delta_0)\left(\frac{R_0}{R_c}\right)^4 - \frac{R_0^2 H_0^2}{R_c^2}\delta_0, \tag{7}$$

as

$$R_c = \sqrt{\frac{1+\delta_0}{\delta_0}} R_0 \simeq \delta_0^{-\frac{1}{2}} R_0, \tag{8}$$

corresponding to the epoch

$$t_c \simeq \frac{t_0}{\delta_0}. \tag{9}$$

The perturbed region must be larger than the Jeans scale, R_J, in order to contract further against the pressure gradient, while it should be smaller than the horizon scale to avoid formation of a separate universe. We thus require

$$R_J \simeq c_s t_c \lesssim R_c \lesssim t_c, \tag{10}$$

or

$$c_s \lesssim \frac{R_c}{t_c} \simeq \frac{R_0}{t_0}\delta_0^{\frac{1}{2}} \lesssim 1, \tag{11}$$

where c_s is the sound velocity equal to $1/\sqrt{3}$ in the radiation dominated era. Since $R_0 \delta_0^{1/2}/t_0$ is time independent, it suffices to calculate the constraint on δ at a specific epoch, say, when the region enters the Hubble radius, $2R = 2t$. We find that the amplitude should lie in the range

$$\frac{1}{3} \lesssim \delta(R=t) \lesssim 1. \tag{12}$$

The gravitational radius, R_g, of the perturbed region in the beginning of contraction with $R_c \simeq c_s t_c$ is given by

$$R_g = 2GM \simeq H^2 R_c^3 \simeq \frac{R_c^3}{t_c^2} \simeq c_s^2 R_c \lesssim R_c. \tag{13}$$

This is somewhat smaller than R_c but it also implies that a black hole will be formed soon after the high density region starts contraction. Thus we expect that a black hole with a mass around a horizon mass at $t = t_c$ will result and it has in fact been shown by numerical calculations (Nadëzhin et al. 1978; Bicknell et al. 1979) that the final mass of the black hole is about $\mathcal{O}(10^{\pm 0.5})$ times the horizon mass at that time. It has been discussed that these black holes do not accrete surrounding matter very much and that their mass do not increase even one order of magnitude (Carr 1975). Note also that evaporation due to the Hawking radiation is unimportant for $M \gg 10^{15}$g (Hawking 1974).

Since the horizon mass at the time t is given by

$$M_{\rm hor} = 10^5 \left(\frac{t}{1\rm sec}\right) M_\odot, \tag{14}$$

what is required in order to produce a significant number of PBHs with mass $M \sim 0.1 M_\odot$ is the sufficient amplitude of density fluctuations on the horizon scale at $t \sim 10^{-6}$sec. Because the initial mass fraction of PBHs, β, is related with the present fraction $\Omega_{\rm BH}$ as

$$\beta = \frac{a(10^{-6}{\rm sec})}{a(t_{\rm eq})}\Omega_{\rm BH} \cong 10^{-8}\Omega_{\rm BH}, \tag{15}$$

where $t_{\rm eq} \sim 10^{10}$sec is the equality time, only an extremely tiny fraction of the universe, $\beta \simeq 10^{-11}$ should collapse into black holes.

That is, the probability of having a density contrast $1/3 \lesssim \delta \lesssim 1$ on the horizon scale at $t \sim 10^{-6}$sec should be equal to β. Let us assume density fluctuations on the relevant scale obey the Gaussian statistics with the dispersion $\overline{\delta}_{\rm BH} \ll 1$, which would be the case in the model introduced in the next sections. Then the probability of PBH formation is estimated as

$$\begin{aligned} \beta &= \int_{1/3}^{1} \frac{1}{\sqrt{2\pi}\,\overline{\delta}_{\rm BH}} \exp\left(-\frac{\delta^2}{2\overline{\delta}_{\rm BH}^2}\right) d\delta \\ &\simeq \int_{1/3}^{1/3+\mathcal{O}(\overline{\delta}_{\rm BH}^2)} \frac{1}{\sqrt{2\pi}\,\overline{\delta}_{\rm BH}} \exp\left(-\frac{\delta^2}{2\overline{\delta}_{\rm BH}^2}\right) d\delta \\ &\simeq \overline{\delta}_{\rm BH} \exp\left(-\frac{1}{18\overline{\delta}_{\rm BH}^2}\right), \end{aligned} \tag{16}$$

which implies that we should have

$$\overline{\delta}_{\rm BH} \simeq 0.05, \tag{17}$$

to produce appropriate amount of PBHs. See figure 1. Although it is true that in principle an exponential accuracy is required on the amplitude of fluctuations in order to produce the desired amount of PBHs, we have not been able to obtain the correspondence between $\overline{\delta}_{\rm BH}$ and $\Omega_{\rm BH}$ with such an accuracy because numerical coefficients appearing in the above expressions, such as 18 in (16), have been calculated based on a rather qualitative argument. We therefore will not attempt exceedingly quantitative analysis in what follows.

Note also that for $\overline{\delta}_{\rm BH} = 0.05$ the threshold of PBH formation, $\delta = 1/3$, corresponds to 6.4 standard deviation. It has been argued by Doroshkevich (1970) that such a high peak has very likely a spherically symmetric shape. Thus the assumption of spherical symmetry in the above discussion is justified and it is also expected that gravitational wave produced during PBH formation is negligibly small.

3 Non-flat perturbation in inflationary cosmology

Since the amplitude of density perturbations on large scales probed by the anisotropy of the background radiation (Smoot et al. 1992) is known to be $\overline{\delta} \simeq 10^{-5}$, the primordial fluctuations must have such a spectral shape that it has an amplitude of 10^{-5} on large scales, sharply increases by a factor of 10^4 on the mass scale of PBHs, and decreases again on smaller scales at the time of horizon crossing. It is difficult to produce such a spectrum of fluctuations in inflationary cosmology with a single component.

In generic inflationary models with a single scalar field ϕ, which drives inflation with a potential $V[\phi]$, the root-mean-square amplitude of adiabatic fluctuations generated is given by

$$\frac{\delta\rho}{\rho}(r) \equiv \overline{\delta}\,(r(\phi)) \cong \frac{8\sqrt{6\pi}V[\phi]^{\frac{3}{2}}}{V'[\phi]M_{Pl}^3}, \tag{18}$$

on the comoving scale $r(\phi)$ when that scale reenters the Hubble radius (Hawking 1982; Starobinsky 1982; Guth & Pi 1982). The right-hand-side is evaluated when the same scale leaves the horizon during inflation. Because of the slow variation of ϕ and rapid cosmic expansion during inflation, (18) implies an almost scale-invariant spectrum in general. Nonetheless one could in principle obtain various shapes of fluctuation spectra making use of the nontrivial dependence of $\overline{\delta}\,(r(\phi))$ on $V[\phi]$ (Hodges & Blumenthal 1990). In order to obtain a desirable spectrum for

PBH formation with a mountain on a particular scale, we must employ a scalar potential with two breaks and a plateau in between (Ivanov et al. 1994). Such a solution is not aesthetically appealing.

Here we instead consider an inflation model with multiple scalar fields in which not only adiabatic but also primordially isocurvature fluctuations are produced. In fact it is much easier to imprint nontrivial structure on the isocurvature spectrum as mentioned in the beginning.

We introduce three scalar fields ϕ_1, ϕ_2, and ϕ_3 in order to generate the desired spectrum of density fluctuation. ϕ_1 is the inflaton field which induces the new inflation (Linde 1982; Albrecht & Steinhardt 1982) with a double-well potential, starting its evolution near the origin where its potential is approximated as

$$U[\phi_1] \cong V_0 - \frac{\lambda_1}{4}\phi_1^4. \tag{19}$$

See Linde (1994) and Vilenkin (1994) for the natural realization of its initial condition. The Hubble parameter during inflation, H_I, is given by

$$H_I^2 = \frac{8\pi V_0}{3M_{Pl}^2}.$$

On the other hand, ϕ_2 is a long-lived scalar field which induces primordially isocurvature fluctuations that contribute to black hole formation later. Finally ϕ_3 is an auxiliary field coupled to both ϕ_1 and ϕ_2 and it changes the effective mass of the latter to imprint a specific feature on the spectrum of its initial fluctuations.

We adopt the following model Lagrangian.

$$\mathcal{L} = -\frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 - \frac{1}{2}(\partial\phi_3)^2 - V[\phi_1, \phi_2, \phi_3] + \mathcal{L}_{\rm int}, \tag{20}$$

where $\mathcal{L}_{\rm int}$ represents interaction of ϕ_j's with other fields. Here $V[\phi_1, \phi_2, \phi_3]$ is the effective scalar potential governing the dynamics of the fields,

$$V[\phi_1, \phi_2, \phi_3] = U[\phi_1] + \frac{\epsilon}{2}(\phi_1^2 - \phi_{1\rm c}^2)\phi_3^3 + \frac{\lambda_3}{4}\phi_3^4 - \frac{\nu}{2}\phi_3^2\phi_2^2 + \frac{\lambda_2}{4}\phi_2^4 + \frac{1}{2}m_2^2\phi_2^2, \tag{21}$$

where λ_j, ϵ, ν, $\phi_{1\rm c}$, and m_2 are positive constants. m_2 is assumed to be much smaller than the scale of inflation, H_I, and it does not affect the dynamics of ϕ_2 during inflation. Hence we ignore it for the moment.

Let us briefly outline how the system evolves before presenting its detailed description. In the early inflationary stage ϕ_1 is smaller than $\phi_{1\rm c}$, and ϕ_3 has its potential minimum off the origin. Then ϕ_2 also settles down to a nontrivial minimum, where it can have an effective mass larger than H_I so that its quantum

fluctuation is suppressed. As ϕ_1 becomes larger than ϕ_{1c}, ϕ_3 rolls down to its origin. Then the potential of ϕ_2 also becomes convex and its amplitude gradually decreases due to its quartic term. However, since its potential is now nearly flat, its motion is extremely slow with its effective mass smaller than H_I well until the end of inflation. In this stage quantum fluctuations are generated to ϕ_2 with a nearly scale-invariant spectrum. Thus the initial spectrum of the isocurvature fluctuations due to ϕ_2 has a scale-invariant spectrum with a cut-off on a large scale.

After inflation, the Hubble parameter starts to decrease in the reheating processes. As it becomes smaller than the effective mass of ϕ_2, the latter starts rapid coherent oscillation. ϕ_2 dissipates its energy in the same way as radiation in the beginning when its oscillation is governed by the quartic term. But later on when $\lambda_2\phi_2^2$ becomes smaller than m_2^2, its energy density decreases more slowly in the same manner as nonrelativistic matter. Thus ϕ_2 contributes to the total energy density more and more later, which implies that the total density fluctuation due to the primordially isocurvature fluctuations or ϕ_2 grows with time. Since what is relevant for PBH formation is the magnitude of fluctuations at the horizon crossing, we thus obtain a spectrum with a larger amplitude on larger scale until the cut-off scale in the initial spectrum is reached, that is, it has a single peak on the mass scale of PBH formation.

In the above scenario we have assumed that ϕ_2 survives until after the PBH formation. On the other hand, were ϕ_2 stable, it would soon dominate the total energy density of the universe in conflict with the successful nucleosynthesis. As a natural possibility we assume that ϕ_2 decays through gravitational interaction, so that it does not leave any unwanted relics with its only trace being the tiny amount of PBHs produced.

In the subsequent sections we describe the detailed evolution of the above model and obtain constraints on the model parameters to produce the right amount of PBHs on the right scale.

4 Background evolution

First we consider the evolution of the homogeneous part of the fields. During inflation, the behavior of the inflaton is governed by the $U[\phi_1]$ part of the potential. Solving the equation of motion with the slow-roll approximation,

$$3H_I\dot{\phi}_1 \cong -U'[\phi_1] \cong \lambda_1\phi_1^3, \tag{22}$$

we find

$$\lambda_1 \phi_1^2(t) = \frac{\lambda_1 \phi_{1i}^2}{1 + \frac{2\lambda_1 \phi_{1i}^2}{3H_I^2} H(t - t_i)}, \tag{23}$$

where ϕ_{1i} is the field amplitude at some initial epoch t_i. The above approximate solution remains valid until $|U''[\phi_1]|$ becomes as large as $9H_I^2$ at $t \equiv t_f$, when inflationary expansion is terminated and we find $\phi_1^2(t_f) = 3H_I^2/\lambda_1$. Then (23) can also be written as

$$\lambda_1 \phi_1^2(t) = \frac{3H_I^2}{2H_I(t_f - t) + 1} \equiv \frac{3H_I^2}{2\tau(t) + 1} \cong \frac{3H_I^2}{2\tau(t)}, \tag{24}$$

where $\tau(t)$ is the e-folding number of exponential expansion after t ($< t_f$) and the last approximation is valid when $\tau \gg 1$. From now on, we often use $\tau(t)$ as a new time variable or to refer to the comoving scale leaving the Hubble radius at t. Note that it is a decreasing function of t.

As stated in the last section, we are taking a view that ϕ_1 determines fate of ϕ_3 and that ϕ_3 controls evolution of ϕ_2 but not vice versa. In order that ϕ_3 does not affect evolution of ϕ_1 the inequality

$$\lambda_1 \lambda_3 \gg \epsilon^2, \tag{25}$$

must be satisfied, while we must have

$$\lambda_2 \lambda_3 \gg \nu^2, \tag{26}$$

so that ϕ_2 does not affect the motion of ϕ_3. We assume these inequalities hold below.

When $\phi_1 < \phi_{1\mathrm{c}}$, both ϕ_2 and ϕ_3 have nontrivial minima which we denote by $\phi_{2\mathrm{m}}$ and $\phi_{3\mathrm{m}}$, respectively. From

$$V_2 = -\nu \phi_3^2 \phi_2 + \lambda_2 \phi_2^3 = 0, \tag{27}$$

and

$$V_3 = \epsilon(\phi_1^2 - \phi_{1\mathrm{c}}^2)\phi_3 + \lambda_3 \phi_3^3 - \nu \phi_2^2 \phi_3 = 0, \tag{28}$$

with $V_j \equiv \partial V / \partial \phi_j$, we find

$$\begin{aligned} \lambda_2 \phi_{2\mathrm{m}}^2(t) &= \nu \phi_{3\mathrm{m}}^2(t) && (29) \\ \lambda_3 \phi_{3\mathrm{m}}^2(t) &= \epsilon \left(\phi_{1\mathrm{c}}^2 - \phi_1^2(t) \right) + \nu \phi_{2\mathrm{m}}^2(t) \cong \epsilon \left(\phi_{1\mathrm{c}}^2 - \phi_1^2(t) \right), && (30) \end{aligned}$$

where (26) was used in the last expression. In the early inflationary stage when $\phi_1 \ll \phi_{1c}$, the effective mass-squared of ϕ_j, V_{jj}, at the potential minimum is given by

$$V_{22}[\phi_1, \phi_{2m}, \phi_{3m}] = 2\lambda_2\phi_{2m}^2 = \frac{2\nu\epsilon}{\lambda_3}(\phi_{1c}^2 - \phi_1^2) \simeq \frac{2\nu\epsilon}{\lambda_3}\phi_{1c}^2 = \frac{3\nu\epsilon}{\lambda_1\lambda_3\tau_c}H_I^2, \tag{31}$$

$$V_{33}[\phi_1, \phi_{2m}, \phi_{3m}] = \frac{3\epsilon}{\lambda_1\tau_c}H_I^2, \tag{32}$$

where τ_c is the epoch when $\phi_1 = \phi_{1c}$. We choose parameters such that

$$\nu\epsilon > \lambda_1\lambda_3\tau_c, \quad \text{and} \quad \epsilon > \lambda_1\tau_c. \tag{33}$$

Then V_{22} and V_{33} are larger than H_I^2 initially at the potential minimum, so that both ϕ_2 and ϕ_3 settle down to $\phi_{2m}(t)$ and $\phi_{3m}(t)$, respectively.

$\phi_{3m}(t)$ decreases down to zero at $\tau = \tau_c$ when $V_{33}[\phi_{3m}]$ also vanishes. Then $V_{33}[\phi_{3m} = 0]$ starts to increase according to ϕ_1 and soon acquires a large positive value, which implies that ϕ_3 practically traces the evolution of $\phi_{3m}(t)$ down to zero without delay. On the other hand, ϕ_2 evolves somewhat differently because it does not acquire a positive effective mass from ϕ_3 at the origin. Although $\phi_2(t)$ traces $\phi_{2m}(t)$ initially, as $V_{22}[\phi_{2m}]$ becomes smaller it can no longer catch up with $\phi_{2m}(t)$. From a generic property of a scaler field with a small mass in the De Sitter background, one can show that this happens when the inequality

$$\left|\frac{1}{\phi_{2m}(t)}\frac{d\phi_{2m}(t)}{dt}\right| > \frac{V_{22}[\phi_{2m}]}{3H}, \tag{34}$$

gets satisfied, or at

$$\tau = \left(1 + \sqrt{\frac{\lambda_1\lambda_3}{2\nu\epsilon}}\right)\tau_c \equiv \tau_l \cong \tau_c, \tag{35}$$

with

$$\phi_2^2(\tau_l) \equiv \phi_{2l}^2 = \sqrt{\frac{\nu\epsilon}{2\lambda_1\lambda_3}}\frac{3H_I^2}{2\lambda_2\tau_l}. \tag{36}$$

Thus ϕ_2 slows down its evolution. In the meantime ϕ_3 vanishes. Then ϕ_2 is governed by the quartic potential. We can therefore summarize its evolution during inflation as

$$\phi_2^2(\tau) \cong \begin{cases} \phi_{2m}^2(\tau) = \frac{3\nu\epsilon H_I^2}{2\lambda_1\lambda_2\lambda_3}\left(\frac{1}{\tau_c} - \frac{1}{\tau}\right), & \tau \gtrsim \tau_l, \\ \phi_{2l}^2\left[1 + \frac{2\lambda_2\phi_{2l}^2}{3H_I^2}(\tau_l - \tau)\right]^{-1}, & 0 < \tau \lesssim \tau_l. \end{cases} \tag{37}$$

Since $\lambda_2\phi_{2l}^2$ is adequately smaller than H_I^2, ϕ_2 remains practically constant in the latter regime. Let us also write down time dependence of its effective mass-squared for later use.

$$V_{22}[\phi_2] \cong \begin{cases} \frac{\nu\epsilon H_I^2}{\lambda_1\lambda_3}\left(\frac{1}{\tau_c}-\frac{1}{\tau}\right), & \tau \gtrsim \tau_l, \\ 3\lambda_2\phi_{2l}^2\left[1+\frac{2\lambda_2\phi_{2l}^2}{3H_I^2}(\tau_l-\tau)\right]^{-1}, & 0<\tau \lesssim \tau_l. \end{cases} \tag{38}$$

5 Generation of fluctuations

We now consider fluctuations in both the scalar fields and metric variables in a consistent manner. We adopt Bardeen's (1980) gauge-invariant variables Φ_A and Φ_H, with which the perturbed metric can be written as

$$ds^2 = -(1+2\Phi_A)dt^2 + a(t)^2(1+2\Phi_H)d\mathbf{x}^2, \tag{39}$$

in the longitudinal gauge. In this gauge scalar field fluctuation $\delta\phi_j$ coincides with the corresponding gauge-invariant variable by itself.

Assuming an $\exp(i\mathbf{kx})$ spatial dependence and working in the Fourier space, the perturbed Einstein and scaler field equations are given by

$$\Phi_A + \Phi_H = 0 \tag{40}$$

$$\dot{\Phi}_H + H\Phi_H = -4\pi G(\dot{\phi}_1\delta\phi_1 + \dot{\phi}_2\delta\phi_2 + \dot{\phi}_3\delta\phi_3) \tag{41}$$

$$\delta\ddot{\phi}_j + 3H\delta\dot{\phi}_j + \left(\frac{k^2}{a^2(t)} + V_{jj}\right)\delta\phi_j = 2V_j\Phi_A + \dot{\Phi}_A\dot{\phi}_j - 3\dot{\Phi}_H\dot{\phi}_j - \sum_{i\neq j} V_{ji}\delta\phi_i, \tag{42}$$

Note that all the fluctuation variables are functions of $\mathbf{k}$ and t.

The above system is quite complicated at a glance. However, using constraints on various model parameters we have obtained so far, *i.e.* (25), (26), and (33), it can somewhat be simplified. First, since ϕ_3 has an effective mass larger than H_I^2 during inflation except in the vicinity of $\tau = \tau_c$, quantum fluctuation on $\delta\phi_3$ is suppressed and moreover its energy density practically vanishes by the end of inflation. We can therefore neglect fluctuations in ϕ_3. On the other hand, we can show that

$$|\dot{\phi}_2| \sim \sqrt{\frac{\nu\epsilon}{\lambda_2\lambda_3}}|\dot{\phi}_1| \ll |\dot{\phi}_1|, \tag{43}$$

with the help of (26) and (33). Hence (41) and (42) with $j=1$ reduce to

$$\dot{\Phi}_H + H\Phi_H = -4\pi G\dot{\phi}_1\delta\phi_1 \tag{44}$$

$$\delta\ddot{\phi}_1 + 3H\delta\dot{\phi}_1 + \left(\frac{k^2}{a^2(t)} + V_{11}\right)\delta\phi_1 = 2V_1\Phi_A - 4\dot{\Phi}_H\dot{\phi}_1. \tag{45}$$

Thus only ϕ_1 contributes to adiabatic fluctuations and it can be calculated in the same manner as in the new inflation model with a single scalar field. This is as expected because ϕ_1 dominates the energy density during inflation. In fact since we are only interested in the growing mode on the super-horizon regime which turns out to be weakly time-dependent as can be seen from the final result, we can consistently neglect time derivatives of metric perturbations and terms with two time derivatives in (44) and (45) during inflation. We thus find

$$\Phi_H = -\Phi_A \cong -\frac{4\pi G}{H_I}\dot{\phi}_1\delta\phi_1. \tag{46}$$

The resultant amplitude of scale-invariant adiabatic fluctuations depends on λ_1 and one can normalize its value using the COBE observation (Smoot et al. 1992) as

$$\lambda_1 = 1.3 \times 10^{-13}. \tag{47}$$

On the other hand, $\delta\phi_2$ satisfies (42) with $j = 2$. From quantum field theory in De Sitter spacetime, it has a root-mean-square amplitude $\delta\phi_2 \cong (H^2/2k^3)^{1/2}$ when the k-mode leaves the Hubble radius if V_{22} is not too large. Since V_2 vanishes when $\phi_2 = \phi_{2\mathrm{m}}$ and $V_2\Phi_A$ remains small even for $\tau \leq \tau_l$, we can neglect all the terms in the right-hand side, to yield

$$\delta\ddot{\phi}_2 + 3H_I\delta\dot{\phi}_2 + V_{22}\delta\phi_2 \cong 0, \tag{48}$$

when $k \ll a(t)H_I$. We can find a WKB solution with the appropriate initial condition,

$$\begin{aligned} \delta\phi_2(k,t) &\cong \sqrt{\frac{H_I^2}{2k^3}}\left(\frac{S(t_k)}{S(t)}\right)^{\frac{1}{2}} \exp\left\{\int_{t_k}^{t}\left[S(t')H_I - \frac{3}{2}H_I\right]dt'\right\}, \\ S(t) &\equiv \frac{3}{2}\sqrt{1 - \frac{4V_{22}}{9H^2}}, \end{aligned} \tag{49}$$

where t_k is the time when k-mode leaves the Hubble radius: $k = a(t_k)H_I$. The above expression is valid when $|\dot{S}| \ll S^2$. In terms of τ, (49) can be expressed as

$$\delta\phi_2(k,\tau) \cong \sqrt{\frac{H_I^2}{2k^3}}\left(\frac{S(\tau_\mathrm{k})}{S(\tau)}\right)^{\frac{1}{2}} \exp\left\{\int_{\tau}^{\tau_\mathrm{k}} S(\tau')d\tau' - \frac{3}{2}(\tau_\mathrm{k} - \tau)\right\}, \tag{50}$$

$$S(\tau) = \begin{cases} \frac{3}{2}\left(1 - \frac{\tau_*}{\tau_c} + \frac{\tau_*}{\tau}\right)^{\frac{1}{2}}, & \tau \gtrsim \tau_l, \\ \frac{3}{2}\left(1 - \frac{2\lambda_2\phi_{2l}^2}{3H_I^2}\left[1 + \frac{2\lambda_2\phi_{2l}^2}{3H_I^2}(\tau_l - \tau)\right]^{-1}\right)^{\frac{1}{2}}, & 0 < \tau \lesssim \tau_l, \end{cases}$$

where $\tau_{\rm k} \equiv \tau(t_k)$ and $\tau_* \equiv \frac{4\nu\epsilon}{3\lambda_1\lambda_3}$. The above equality is valid until the end of inflation at $t = t_f$ or $\tau = 0$.

6 Evolution of the universe after inflation

Let us assume the universe is rapidly and efficiently reheated at $t = t_f$ for simplicity to avoid further complexity. (see, *e.g.* Kofman et al. (1994), Shtanov et al. (1995), and Boyanovsky et al. (1995) for recent discussion on efficient reheating.) Then the reheat temperature is given by

$$T_R \cong 0.1\sqrt{H_I M_{Pl}}. \tag{51}$$

If there is no further significant entropy production later, one can calculate the epoch, $\tau(L)$, when the comoving length scale corresponding to L pc today left the Hubble horizon during inflation as

$$\tau(L) = 37 + \ln\left(\frac{L}{1\,\text{pc}}\right) + \frac{1}{2}\ln\left(\frac{H_I}{10^{10}\text{ GeV}}\right). \tag{52}$$

Then the comoving horizon scale at $t = 10^{-6}$sec, or $L = 0.03$ pc corresponds to $\tau \cong 34 \equiv \tau_{\rm m}$ and the present horizon scale $\simeq 3000$Mpc to $\tau \cong 59$.

On the other hand, $\phi_2(t)$ and $\delta\phi_2(\mathbf{k}, t)$ evolve according to

$$\begin{aligned} \ddot{\phi}_2 + 3H\dot{\phi}_2 + \lambda_2\phi_2^3 + m_2^2\phi_2 &= 0, & (53) \\ \delta\ddot{\phi}_2 + 3H\delta\dot{\phi}_2 + (3\lambda_2\phi_2^2 + m_2^2)\delta\phi_2 &\cong 0, & (54) \end{aligned}$$

where the Hubble parameter is now time-dependent: $H = 1/2t$, and the latter equation is valid for $k \ll aH$.

When H^2 becomes smaller than $\lambda_2\phi_2^2$ ($\gg m_2^2$), both ϕ_2 and $\delta\phi_2$ start rapid oscillation around the origin. Using (50) one can express the amplitude of the gauge-invariant comoving fractional density perturbation of ϕ_2, Δ_2, as

$$\Delta_2 = \frac{1}{\rho_2}\left(\dot{\phi}_2\delta\dot{\phi}_2 - \ddot{\phi}_2\delta\phi_2 - \dot{\phi}_2{}^2\Phi_A\right) \simeq 4\frac{\delta\phi_2}{\phi_2}\bigg|_{\tau=0} \equiv 4\frac{\delta\phi_{2\rm f}}{\phi_{2\rm f}}, \tag{55}$$

in the beginning of oscillation. Here

$$\rho_2 \equiv \frac{1}{2}\dot{\phi_2}^2 + \frac{\lambda_2}{4}\phi_2^4 + \frac{1}{2}m_2^2\phi_2^2 \tag{56}$$

is the energy density of ϕ_2.

Using the virial theorem one can easily show that it decreases in proportion to $a^{-4}(t)$ as long as $\lambda_2\phi_2^2 \gtrsim m_2^2$. Thus the amplitude of ϕ_2 decreases with $a^{-1}(t)$. On the other hand, $\delta\phi_2$ has a rapidly oscillating mass term when $\lambda_2\phi_2^2 \gtrsim m_2^2$, which causes parametric amplification. We have numerically solved equations (53) and (54) with various initial conditions with $|\phi_2| \gg |\delta\phi_2|$ initially. We have found that in all cases the amplitude of $\delta\phi_2$ remains constant as long as m_2 is negligible. Thus Δ_2 increases in proportion to $t^{1/2}$ in this regime and Δ_2 becomes as large as

$$\Delta_2 \cong 4\frac{\delta\phi_{2\mathrm{f}}}{\phi_{2\mathrm{f}}}\left(\frac{\lambda_2\phi_{2\mathrm{f}}^2}{m_2^2}\right)^{\frac{1}{2}}, \tag{57}$$

while the ratio of ρ_2 to the total energy density, ρ_{tot}, which is now dominated by radiation, remains constant:

$$\frac{\rho_2}{\rho_{\mathrm{tot}}} = 2\pi\left(\frac{\phi_{2\mathrm{f}}}{M_{Pl}}\right)^2. \tag{58}$$

As $\lambda_2\phi_2^2$ becomes smaller than m_2^2, ϕ_2 and $\delta\phi_2$ come to satisfy the same equation of motion, see (53) and (54), and Δ_2 saturates to the constant value (57). At the same time ρ_2 starts to decrease less rapidly than radiation, in proportion to $a^{-3}(t)$. Since Δ_2 contributes to the total comoving density fluctuation by the amplitude

$$\Delta \cong \frac{\rho_2}{\rho_{\mathrm{tot}}}\Delta_2, \tag{59}$$

it increases in proportion to $a(t) \propto t^{1/2}$. In the beginning of this stage, we find $H^2 \cong m_2^4/\lambda_2\phi_{2\mathrm{f}}^2$, to yield

$$\Delta \cong 8\pi\frac{\delta\phi_{2\mathrm{f}}}{\phi_{2\mathrm{f}}}\left(\frac{\lambda_2\phi_{2\mathrm{f}}^2}{m_2^2}\right)^{\frac{1}{2}}\left(\frac{\phi_{2\mathrm{f}}}{M_{Pl}}\right)^2\left(\frac{2m_2^2 t}{\sqrt{\lambda_2}\phi_{2\mathrm{f}}}\right)^{\frac{1}{2}}, \tag{60}$$

at a later time t.

In order to relate it with the initial condition required for PBH formation, we must estimate it at the time k-mode reenters the Hubble radius, t_k^*, defined by

$$k = 2\pi a(t_k^*)H(t_k^*) = \frac{\pi a_f}{(t_k^* t_f)^{\frac{1}{2}}}. \tag{61}$$

Since k can also be expressed as $k = 2\pi a_f e^{-\tau_\mathrm{k}} H_I$, the amplitude of comoving density fluctuation at $t = t_k^*$ is given by

$$\Delta(k, t_k^*) \cong 8\pi \frac{\delta\phi_{2\mathrm{f}}}{\phi_{2\mathrm{f}}} \left(\frac{\sqrt{\lambda_2}\phi_{2\mathrm{f}}}{H_I}\right)^{\frac{1}{2}} \left(\frac{\phi_{2\mathrm{f}}}{M_{Pl}}\right)^2 e^{\tau_\mathrm{k}}. \tag{62}$$

7 Constraints on model parameters

In §2 we discussed the necessary condition on the amplitude of fluctuations for PBH formation using the uniform Hubble constant gauge. Hence we should calculate the predicted amplitude in this gauge, which is a linear combination of Δ and the gauge-invariant velocity perturbation. However, in the present case in which Δ grows in proportion to $a(t)$ in the radiation-dominant universe, one finds that the latter quantity vanishes and that density fluctuation in the uniform Hubble constant gauge coincides with Δ (Kodama & Sasaki 1984). Thus we finally obtain the quantity to be compared with $\overline{\delta}_\mathrm{BH}$ in (16), namely, the root-mean-square amplitude of density fluctuation on scale $r = 2\pi/k$ at the horizon crossing, $\overline{\delta}(r)$, as

$$\overline{\delta}(r) = \left[\frac{4\pi k^3}{(2\pi)^3}|\Delta(k, t_k^*)|^2\right]^{\frac{1}{2}} \cong 4\left(\frac{\sqrt{\lambda_2}H_I\phi_{2\mathrm{f}}^3}{M_{Pl}^4}\right)^{\frac{1}{2}} e^{\tau_\mathrm{k}} C_f(\tau_\mathrm{k}, \tau_\mathrm{c}, \tau_*), \tag{63}$$

with

$$C_f(\tau_\mathrm{k}, \tau_\mathrm{c}, \tau_*) \equiv \left(\frac{S(\tau_\mathrm{k})}{S(0)}\right)^{\frac{1}{2}} \exp\left\{\int_0^{\tau_\mathrm{k}} S(\tau')d\tau' - \frac{3}{2}\tau_\mathrm{k}\right\}, \tag{64}$$

where we have used (50). We also find

$$\phi_{2\mathrm{f}}^2 = \sqrt{\frac{3\tau_*}{8}}\left(1 + \sqrt{\frac{2}{3\tau_*}}\right)^{-1}\left(1 + \sqrt{\frac{3\tau_*}{8}}\right)^{-1}\frac{3H_I^2}{2\lambda_2\tau_\mathrm{c}}, \tag{65}$$

from (36) and (37).

The remaining task is to choose values of parameters so that $\overline{\delta}(r)$ has a peak on the comoving horizon scale at $t = 10^{-6}$sec, which we denote by r_m, corresponding to $\tau_\mathrm{k} = \tau_\mathrm{m} \cong 34$, with its amplitude $\overline{\delta}(r_\mathrm{m}) \cong 0.05$. We thus require

$$\begin{aligned} \frac{d\ln\overline{\delta}(r)}{d\tau_\mathrm{k}} &= \frac{S'(\tau_\mathrm{k})}{2S(\tau_\mathrm{k})} + S(\tau_\mathrm{k}) - \frac{1}{2} \\ &= -\frac{\tau_*\tau_\mathrm{c}}{4\tau_\mathrm{k}\left[\tau_\mathrm{c}\tau_\mathrm{k} + \tau_*\left(\tau_\mathrm{c} - \tau_\mathrm{k}\right)\right]} + \frac{3}{2}\left(1 - \frac{\tau_*}{\tau_\mathrm{c}} + \frac{\tau_*}{\tau_\mathrm{k}}\right)^{\frac{1}{2}} - \frac{1}{2} \end{aligned} \tag{66}$$

vanishes at $\tau_{\rm k} = \tau_{\rm m}$, which gives us a relation between $\tau_{\rm c}$ and τ_*. Since $\tau_{\rm c}$ roughly corresponds to the comoving scale where scale-invariance of primordial fluctuation Δ_2 is broken, the peak at $\overline{\delta}(r_{\rm m})$ becomes the sharper, the closer $\tau_{\rm c}$ approaches $\tau_{\rm m}$.

For example, if we take $\tau_{\rm c} = 30$ we find

$$\tau_* = \frac{4\nu\epsilon}{3\lambda_1\lambda_2} = 200, \tag{67}$$

so that

$$C_f = 0.13 \quad \text{and} \quad \phi_{2{\rm f}}^2 = 0.045\frac{H_I^2}{\lambda_2}. \tag{68}$$

In order to have $\overline{\delta}(r_{\rm m}) = 0.05$, we find

$$\frac{1}{\sqrt{\lambda_2}}\left(\frac{H_I}{M_{Pl}}\right)^2 = 1.7 \times 10^{-15}, \tag{69}$$

which can easily be satisfied with some reasonable choices of λ_2 and H_I. However, it is not the final constraint. Since we are assuming that the universe is dominated by radiation at this time, we require

$$\frac{\rho_2}{\rho_{\rm tot}} = 2\pi\left(\frac{\phi_{2{\rm f}}}{M_{Pl}}\right)^2\left(\frac{m_2^2}{\sqrt{\lambda_2}\phi_{2{\rm f}}H_I}\right)^{\frac{1}{2}} e^{\tau_{\rm m}} \ll 1. \tag{70}$$

Furthermore ϕ_2 should decay some time after $t = 10^{-6}$sec so as not to dominate the energy density of the universe which would hamper the primordial nucleosynthesis. Assuming that it decays only through gravitational interaction, its life time is given by

$$\tau_{\phi_2} \cong \frac{M_{Pl}^2}{m_2^3} = 10^{-5.5}\left(\frac{m_2}{10^{6.5}{\rm GeV}}\right)^{-3} {\rm sec}\,. \tag{71}$$

Now we have displayed all the necessary equalities and inequalities the model parameters should satisfy. Since there is a wide range of allowed region in the multi-dimensional space of parameters, we do not work out the details of the constraints but simply give one example of their values with which all the requirements are satisfied:

$$\begin{aligned}
H_I &= 1.7 \times 10^{10}{\rm GeV}, \\
m_2 &= 3.2 \times 10^{6}{\rm GeV}, \\
\lambda_1 &= 1.3 \times 10^{-13}, \\
\lambda_2 &= 1.4 \times 10^{-6}, \\
\lambda_3 &= \nu = 6.7 \times 10^{-8}, \\
\epsilon &= 2.0 \times 10^{-11},
\end{aligned} \tag{72}$$

for which $\rho_2/\rho_{\rm tot} = 0.1$ at $t = 10^{-6}$sec and inequalities (25) and (26) are maximally satisfied.

In figure 2 we have depicted the qualitative mass spectrum of produced PBHs for different values of $\tau_{\rm c}$ where

$$\beta(M) = \overline{\delta}(r) \exp\left(-\frac{1}{18\overline{\delta}(r)^2}\right) \tag{73}$$

has been shown as a function of the horizon mass when the scale r reenters the horizon.

8 Conclusion

In the present paper we have considered possibility to produce a significant amount of PBHs on a specific mass scale by generating appropriate spectrum of density fluctuations in inflationary cosmology. We have reached a model with the desired feature making use of a simple polynomial potential (21) without introducing any break in the potential of the scalar fields. We have chosen values of the model parameters so that these PBHs can account for the observed MACHOs. In order to set the order of magnitude of the mass scale of the black holes and that of their abundance correctly, we had to tune some combinations of model parameters such as (67) and (69) with two digits' accuracy. However, there exists a wide range of allowed region in the parameter space to realize it. We also note that the precise values such as those quoted in (72) are not of much significance, primarily because the formula for the fraction of PBHs (16) is only a qualitative one. In this respect we have restricted ourselves to the analytic treatment of evolution of fluctuation, which is not exponentially accurate. In the event a more precise formula for PBH fraction is obtained, full numerical analysis of fluctuations would be required. At the present stage, however, analytic treatment is more appropriate with which dependence of the results on physical parameters are understood more clearly.

It is evident that our model can be applicable to produce PBHs with a different mass and abundance by slightly changing values of parameters. For example, we could produce black holes with mass $\sim 10^6 M_\odot$ which would act as a central engine of AGNs. These black holes are usually considered to have formed in the post-recombination universe (Loeb 1993), but they might have formed in the early universe at $t \sim 10$sec corresponding to the onset of primordial nucleosynthesis. Note that this would not hamper successful nucleosynthesis because the root-mean-square amplitude of density fluctuation required for such black hole formation is

still much smaller than unity.

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Figure captions

Figure 1 Fraction of primordial black holes as a function of root-mean-square amplitude of density fluctuations (eq.[16]) Gaussian distribution of fluctuations is assumed.

Figure 2 Expected mass spectrum of primordial black holes with different values of $\tau_{\rm c}$.